CHAPTER 2

CONIC SECTIONS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Determine the equation of a curve using the locus of the equation.

2. Determine the equation and properties of a circle, a parabola, an ellipse, and a hyperbola.

3. Transform polar coordinates to Cartesian coordinates and vice-versa.

INTRODUCTION

This chapter is a continuation of the study of analytic geometry. The figures presented in this chapter are plane figures, which are included in the general class of conic sections or simply "conics."

Conic sections are so named because they are all plane sections of a right circular cone. A circle is formed when a cone is cut perpendicular to its axis. An ellipse is produced when the cone is cut obliquely to the axis and the surface. A hyperbola results when the cone is intersected by a plane parallel to the axis, and a parabola results when the intersecting plane is parallel to an element of the surface. These are illustrated in figure 2-1.

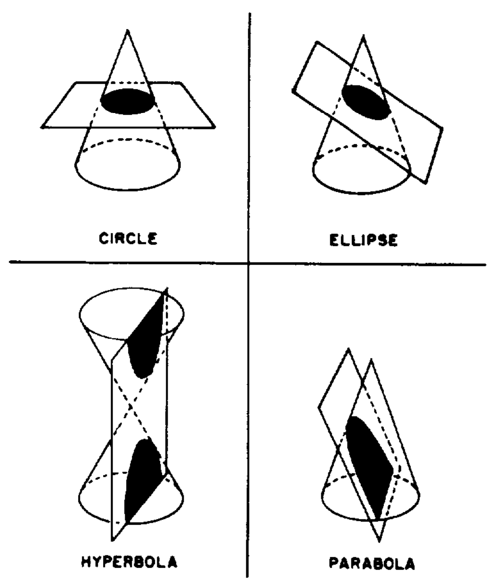

Figure 2-1.—Conic sections.

When such a curve is plotted on a coordinate system, it may be defined as follows:

A *conic section* is the locus of all points in a plane whose distance from a fixed point is a constant ratio to its distance from a fixed line. The fixed point is the *focus*, and the fixed line is the *directrix*.

The ratio referred to in the definition is called the *eccentricity* (e). If the eccentricity is greater than 0 and less than 1, the curve is an ellipse. If e is greater than 1, the curve is a hyperbola. If e is equal to 1, the curve is a parabola. A circle is a special case having an eccentricity equal to 0. It is actually a limiting case of an ellipse in which the eccentricity approaches 0. Thus, if

$0 < e < 1$, it is an *ellipse*;

$e > 1$, it is a *hyperbola*;

$e = 1$, it is a *parabola*;

$e = 0$, it is a *circle*.

The eccentricity, focus, and directrix are used in the algebraic analysis of conic sections and their corresponding equations. The concept of the locus of an equation also enters into analytic geometry; this concept is discussed before the individual conic sections are presented.

THE LOCUS OF AN EQUATION

In chapter 1 of this course, methods for analysis of linear equations are presented. If a group of x and y values [or ordered pairs, $P(x,y)$] that satisfy a given linear equation are plotted on a coordinate system, the resulting graph is a straight line.

When higher-ordered equations such as

$$x^2 + y^2 = 1 \text{ or } y = \sqrt{2x + 3}$$

are encountered, the resulting graph is not a straight line. However, the points whose coordinates satisfy most of the equations in x and y are normally not scattered in a random field. If the values are plotted, they will seem to follow a line or curve (or a combination of lines and curves). In many texts the plot of an equation is called a curve, even when it is a straight line. This curve is called the *locus* of the equation. The *locus* of an equation is a curve containing those points, and only those points, whose coordinates satisfy the equation.

At times the curve may be defined by a set of conditions rather than by an equation, though an equation may be derived from the given conditions. Then the curve in question would be the locus of all points that fit the conditions. For instance a circle may be said to be the locus of all points in a plane that is a fixed distance from a fixed point. A straight line may be defined as the locus of all points in a plane equidistant from two fixed points. The method of expressing a set of conditions in analytical form gives an equation. Let us draw up a set of conditions and translate them into an equation.

EXAMPLE: What is the equation of the curve that is the locus of all points equidistant from the two points (5,3) and (2,1)?

SOLUTION: First, as shown in figure 2-2, choose some point having coordinates (x,y). Recall from chapter 1 of this course that the distance between this point and (2,1) is given by

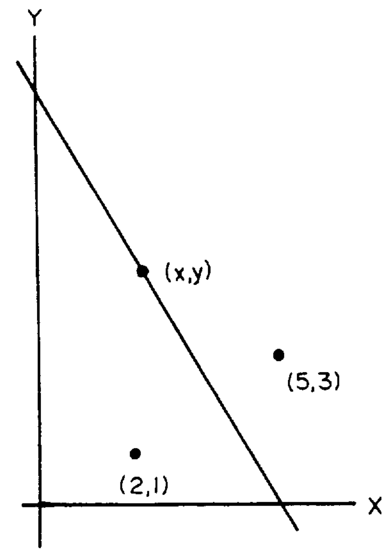

Figure 2-2.—Locus of points equidistant from two given points.

$$\sqrt{(x-2)^2 + (y-1)^2}$$

The distance between point (x,y) and (5,3) is given by

$$\sqrt{(x-5)^2 + (y-3)^2}$$

Equating these distances, since the point is to be equidistant from the two given points, we have

$$\sqrt{(x-2)^2 + (y-1)^2} = \sqrt{(x-5)^2 + (y-3)^2}$$

Squaring both sides, we have

$$(x-2)^2 + (y-1)^2 = (x-5)^2 + (y-3)^2$$

Expanding, we have

$$x^2 - 4x + 4 + y^2 - 2y + 1$$

$$= x^2 - 10x + 25 + y^2 - 6y + 9$$

Canceling and collecting terms, we see that

$$4y + 5 = -6x + 34$$

$$4y = -6x + 29$$

$$y = -1.5x + 7.25$$

This is the equation of a straight line with a slope of minus 1.5 and a y intercept of $+7.25$.

EXAMPLE: Find the equation of the curve that is the locus of all points equidistant from the line $x = -3$ and the point (3,0).

SOLUTION: As shown in figure 2-3, the distance from the point (x,y) on the curve to the line $x = -3$ is

$$\sqrt{[x-(-3)]^2 + (y-y)^2} = \sqrt{(x+3)^2}$$

The distance from the point (x,y) to the point (3,0) is

$$\sqrt{(x-3)^2 + (y-0)^2}$$

Equating the two distances yields

$$\sqrt{(x+3)^2} = \sqrt{(x-3)^2 + y^2}$$

Squaring and expanding both sides yields

$$x^2 + 6x + 9 = x^2 - 6x + 9 + y^2$$

Canceling and collecting terms yields

$$y^2 = 12x$$

which is the equation of a parabola.

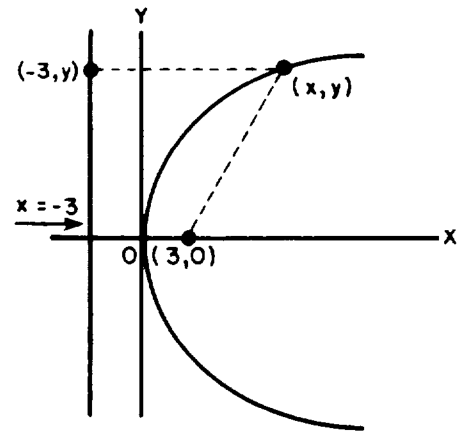

Figure 2-3.—Parabola.

EXAMPLE: What is the equation of the curve that is the locus of all points in which the ratio of its distance from the point (3,0) to its distance from the line $x = 25/3$ is equal to 3/5? Refer to figure 2-4.

SOLUTION: The distance from the point (x,y) to the point (3,0) is given by

$$d_1 = \sqrt{(x-3)^2 + (y-0)^2}$$

The distance from the point (x,y) to the line $x = 25/3$ is

$$d_2 = \frac{25}{3} - x$$

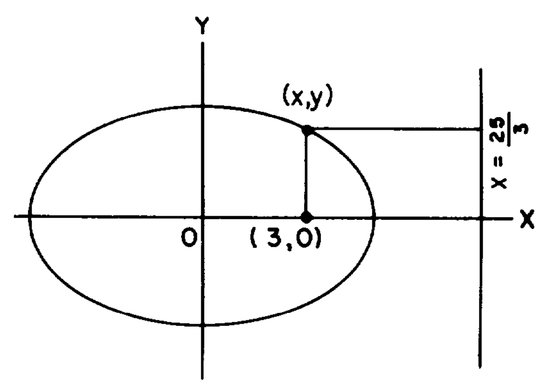

Figure 2-4.—Ellipse.

Since

$$\frac{d_1}{d_2} = \frac{3}{5} \text{ or } d_1 = \frac{3}{5}d_2$$

then

$$\sqrt{(x-3)^2 + y^2} = \frac{3}{5}\left(\frac{25}{3} - x\right)$$

Squaring both sides and expanding, we have

$$x^2 - 6x + 9 + y^2 = \frac{9}{25}\left(x^2 - \frac{50}{3}x + \frac{625}{9}\right)$$

$$x^2 - 6x + 9 + y^2 = \frac{9}{25}x^2 - 6x + 25$$

Collecting terms and transposing, we see that

$$\frac{16}{25}x^2 + y^2 = 16$$

Dividing both sides by 16, we have

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

This is the equation of an ellipse.

PRACTICE PROBLEMS:

Find the equation of the curve that is the locus of all points equidistant from the following:

1. The points (0,0) and (5,4).
2. The points (3, −2) and (−3,2).
3. The line $x = -4$ and the point (3,4).
4. The point (4,5) and the line $y = 5x - 4$.

   HINT: Use the standard distance formula to find the distance from the point $P(x,y)$ and the point $P(4,5)$. Then use the formula for finding the distance from a point to a line, given in chapter 1 of this course, to find the distance from $P(x,y)$ to the given line. Put the equation of the line in the form $Ax + By + C = 0$.

ANSWERS:

1. $10x + 8y - 41 = 0$ or $y = -1.25x + \frac{41}{8}$

2. $2y = 3x$ or $y = \frac{3}{2}x$

3. $y^2 - 8y = 14x - 9$ or $(y - 4)^2 = 14x + 7$

4. $x^2 + 10xy + 25y^2 - 168x - 268y + 1050 = 0$

THE CIRCLE

A *circle* is the locus of all points, in a plane that is a fixed distance from a fixed point, called the *center*.

The fixed distance spoken of here is the *radius* of the circle.

The equation of a circle with its center at the origin (fig. 2-5) is

$$\sqrt{(x - 0)^2 + (y - 0)^2} = r$$

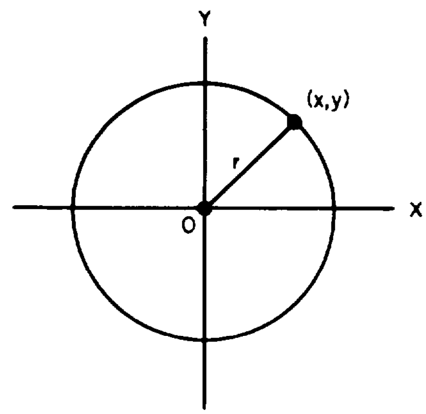

Figure 2-5.—Circle with center at the origin.

where (x,y) is a point on the circle and r is the radius (r replaces d in the standard distance formula). Then

$$\sqrt{x^2 + y^2} = r$$

or

$$x^2 + y^2 = r^2 \qquad (2.1)$$

If the center of a circle, figure 2-6, is at some point where $x = h$, $y = k$, then the distance of the point (x,y) from the center will be constant and equal to

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or

$$(x - h)^2 + (y - k)^2 = r^2 \qquad (2.2)$$

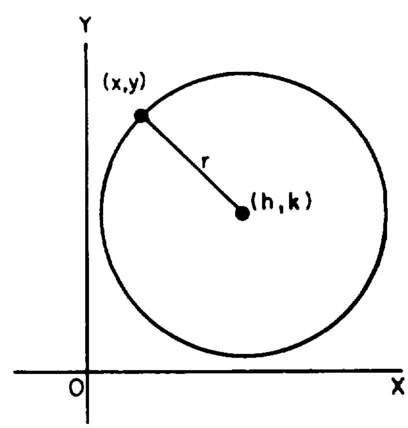

Figure 2-6.—Circle with center at (h,k).

Equations (2.1) and (2.2) are the standard forms for the equation of a circle. Equation (2.1) is merely a special case of equation (2.2) in which h and k are equal to zero.

The equation of a circle may also be expressed in the general form:

$$x^2 + y^2 + Bx + Cy + D = 0 \tag{2.3}$$

where B, C, and D are constants.

Theorem: *An equation of the second degree in which the coefficients of the* x^2 *and* y^2 *terms are equal and the xy term does not exist, represents a circle.*

Whenever we find an equation in the form of equation (2.3), we should convert it to the form of equation (2.2) so that we have the coordinates of the center of the circle and the radius as part of the equation. This may be done as shown in the following example problems:

EXAMPLE: Find the coordinates of the center and the radius of the circle described by the following equation:

$$x^2 + y^2 - 4x - 6y - 3 = 0$$

SOLUTION: First rearrange the terms

$$x^2 - 4x + y^2 - 6y - 3 = 0$$

and complete the square in both x and y. Completing the square is discussed in the chapter on quadratic solutions in *Mathematics*, Volume 1. The procedure consists basically of adding certain quantities to both sides of a second-degree equation to form the sum of two perfect squares. When both the first- and second-degree members are known, the square of one-half the coefficient of the first-degree term is added to both sides of the equation. This will allow the quadratic equation to be factored into the sum of two perfect squares. To complete the square in x in the given equation

$$x^2 - 4x + y^2 - 6y - 3 = 0$$

add the square of one-half the coefficient of x to both sides of the equation

$$x^2 - 4x + (2)^2 + y^2 - 6y - 3 = 0 + (2)^2$$

then

$$(x^2 - 4x + 4) + y^2 - 6y - 3 = 4$$

$$(x - 2)^2 + y^2 - 6y - 3 = 4$$

completes the square in x.

If we do the same for y,

$$(x - 2)^2 + y^2 - 6y + (3)^2 - 3 = 4 + (3)^2$$

$$(x - 2)^2 + (y^2 - 6y + 9) - 3 = 4 + 9$$

$$(x - 2)^2 + (y - 3)^2 - 3 = 4 + 9$$

completes the square in y.

Transpose all constant terms to the right-hand side and simplify:

$$(x - 2)^2 + (y - 3)^2 = 4 + 9 + 3$$

$$(x - 2)^2 + (y - 3)^2 = 16$$

The equation is now in the standard form of equation (2.2). This equation represents a circle with the center at (2,3) and with a radius equal to $\sqrt{16}$ or 4.

EXAMPLE: Find the coordinates of the center and the radius of the circle given by the equation

$$x^2 + y^2 + \frac{1}{2}x - 3y - \frac{27}{16} = 0$$

SOLUTION: Rearrange and complete the square in both x and y:

$$x^2 + \frac{1}{2}x + y^2 - 3y - \frac{27}{16} = 0$$

$$\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \left(y^2 - 3y + \frac{9}{4}\right) - \frac{27}{16} = \frac{1}{16} + \frac{9}{4}$$

Transposing all constant terms to the right-hand side and adding, results in

$$\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \left(y^2 - 3y + \frac{9}{4}\right) = 4$$

Reducing to the equation in standard form results in

$$\left(x + \frac{1}{4}\right)^2 + \left(y - \frac{3}{2}\right)^2 = (2)^2$$

Thus, the equation represents a circle with its center at $(-1/4, 3/2)$ and a radius equal to 2.

PRACTICE PROBLEMS:

Find the coordinates of the center and the radius for the circles described by the following equations:

1. $x^2 - \frac{4}{5}x + y^2 - 4y + \frac{29}{25} = 0$

2. $x^2 + 6x + y^2 - 14y = 23$

3. $x^2 - 14x + y^2 + 22y = -26$

4. $x^2 + y^2 + \frac{2}{5}x + \frac{2}{3}y = \frac{2}{25}$

5. $x^2 + y^2 - 1 = 0$

ANSWERS:

1. Center $\left(\frac{2}{5}, 2\right)$, radius $\sqrt{3}$

2. Center $(-3,7)$, radius 9

3. Center $(7,-11)$, radius 12

4. Center $\left(-\frac{1}{5}, -\frac{1}{3}\right)$, radius $\frac{2\sqrt{13}}{15}$

5. Center (0,0), radius 1

In certain situations you will want to consider the following general form of a circle

$$x^2 + y^2 + Bx + Cy + D = 0$$

as the equation of a circle in which the specific values of the constants B, C, and D are to be determined. In this problem the unknowns to be found are not x and y, but the values of the constants B, C, and D. The conditions that define the circle are used to form algebraic relationships between these constants. For example, if one of the conditions imposed on the circle is that it pass through the point (3,4), then the general form is written with x and y replaced by 3 and 4, respectively; thus,

$$x^2 + y^2 + Bx + Cy + D = 0$$

is rewritten as

$$(3)^2 + (4)^2 + B(3) + C(4) + D = 0$$

$$3B + 4C + D = -25$$

Three independent constants (B, C, and D) are in the equation of a circle; therefore, three conditions must be given to define a circle. Each of these conditions will yield an equation with B, C, and D as the unknowns. These three equations are then solved simultaneously to determine the values of the constants, which satisfy all of the equations. In an analysis, the number of independent constants in the general equation of a curve indicate how many conditions must be set before a curve can be completely defined. Also, the number of unknowns in an equation indicates the number of equations that must be solved simultaneously to find the values of the unknowns. For example, if B, C, and D are unknowns in an equation, three separate equations involving these variables are required for a solution.

A circle may be defined by three noncollinear points; that is, by three points not lying on a straight line. Only one circle is possible through any three noncollinear points. To find the *equation of the circle determined by three points, substitute the x and y values of each of the given points into the general equation to form three equations with B, C, and D as the unknowns. These equations are then solved simultaneously to find the values of B, C, and D in the equation which satisfies the three given conditions.*

The solution of simultaneous equations involving two variables is discussed in *Mathematics,* Volume 1. Systems involving three variables use an extension of the same principles, but with three equations instead of two. Step-by-step explanations of the solution are given in the example problems.

EXAMPLE: Write the equation of the circle that passes through the points (2,8), (5,7), and (6,6).

SOLUTION: The method used in this solution corresponds to the addition-subtraction method used for solution of equations involving two variables. However, the method or combination of methods used depends on the particular problem. No single method is best suited to all problems.

First, write the general form of a circle:

$$x^2 + y^2 + Bx + Cy + D = 0$$

For each of the given points, substitute the given values for x and y and rearrange the terms:

For (2,8) $\quad 4 + 64 + 2B + 8C + D = 0$

$$2B + 8C + D = -68$$

For (5,7) $\quad 25 + 49 + 5B + 7C + D = 0$

$$5B + 7C + D = -74$$

For (6,6) $\quad 36 + 36 + 6B + 6C + D = 0$

$$6B + 6C + D = -72$$

To aid in the explanation, we number the three resulting equations:

$$2B + 8C + D = -68 \qquad \text{Equation (1)}$$

$$5B + 7C + D = -74 \qquad (2)$$

$$6B + 6C + D = -72 \qquad (3)$$

The first step is to eliminate one of the unknowns and have two equations and two unknowns remaining. The coefficient of D is the same in all three equations and is, therefore, the one most easily eliminated by addition and subtraction. To eliminate D, subtract equation (2) from equation (1):

$$\begin{array}{rlr} 2B + 8C + D = -68 & & (1) \\ \underline{5B + 7C + D = -74} & & (-)\ (2) \\ -3B + C = 6 & & (4) \end{array}$$

Subtract equation (3) from equation (2):

$$\begin{array}{rlr} 5B + 7C + D = -74 & & (2) \\ \underline{6B + 6C + D = -72} & & (-)\ (3) \\ -B + C = -2 & & (5) \end{array}$$

We now have two equations, (4) and (5), in two unknowns that can be solved simultaneously. Since the coefficient of C is the same in both equations, it is the most easily eliminated variable.

To eliminate C, subtract equation (4) from equation (5):

$$\begin{array}{rlr} -B + C = -2 & & (5) \\ \underline{-3B + C = 6} & & (-)\ (4) \\ 2B = -8 & & \\ B = -4 & & (6) \end{array}$$

To find the value of C, substitute the value found for B in equation (6) in equation (4) or (5)

$$\begin{array}{rlr} -B + C = -2 & & \text{using (5)} \\ -(-4) + C = -2 & & \\ C = -6 & & (7) \end{array}$$

Now the values of B and C can be substituted in any one of the original equations to determine the value of D.

If the values are substituted in equation (1),

$$\begin{array}{rlr} 2B + 8C + D = -68 & & (1) \\ 2(-4) + 8(-6) + D = -68 & & \\ -8 - 48 + D = -68 & & \\ D = -68 + 56 & & \\ D = -12 & & (8) \end{array}$$

The solution of the system of equations gave values for three independent constants in the general equation

$$x^2 + y^2 + Bx + Cy + D = 0$$

When the constant values are substituted, the equation takes the form of

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

Now rearrange and complete the square in both x and y:

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) - 12 = 4 + 9$$

$$(x - 2)^2 + (y - 3)^2 = 25$$

The equation now corresponds to a circle with its center at (2,3) and a radius of 5. This is the circle passing through three given points, as shown in figure 2-7, view A.

The previous example problem showed one method we can use to determine the equation of a circle when three points are given. The next example shows another method we can use to solve the same problem. One of the most important things to keep in mind when you study analytic geometry is that many problems may be solved by more than one method. Each problem should be analyzed carefully to determine what relationships exist between the given data and the desired results of the problem. Relationships such as distance from one point to another, distance from a point to a line, slope of a line, and the Pythagorean theorem will be used to solve various problems.

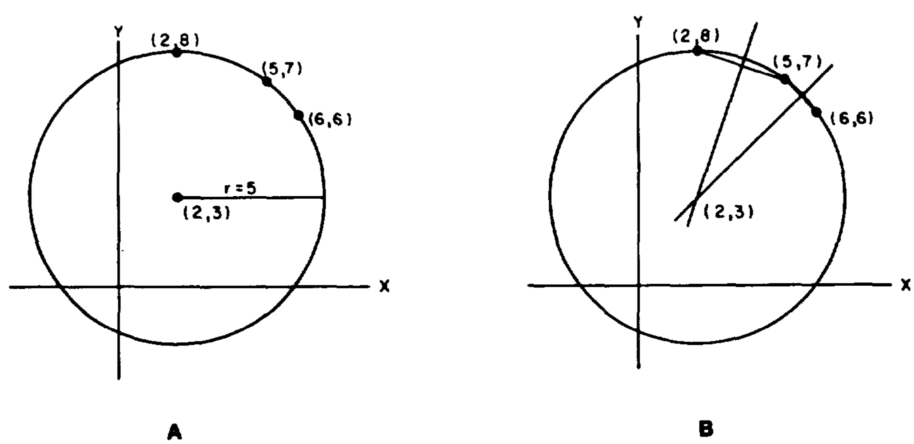

Figure 2-7.—Circle described by three points.

EXAMPLE: Find the equation of the circle that passes through the points (2,8), (5,7), and (6,6). Use a method other than that used in the previous example problem.

SOLUTION: A different method of solving this problem results from the reasoning in the following paragraphs:

The center of the desired circle will be the intersection of the perpendicular bisectors of the chords connecting points (2,8) with (5,7) and (5,7) with (6,6), as shown in figure 2-7, view B.

The perpendicular bisector of the line connecting two points is the locus of all points equidistant from the two points. Using this analysis, we can get the equations of the perpendicular bisectors of the two lines.

Equating the distance formulas that describe the distances from the center, point (x,y), which is equidistant from the points (2,8) and (5,7), gives

$$\sqrt{(x-2)^2+(y-8)^2} = \sqrt{(x-5)^2+(y-7)^2}$$

Squaring both sides gives

$$(x-2)^2+(y-8)^2 = (x-5)^2+(y-7)^2$$

or

$$x^2-4x+4+y^2-16y+64 =$$

$$x^2-10x+25+y^2-14y+49$$

Canceling and combining terms results in

$$6x-2y=6$$

or

$$3x-y=3$$

Follow the same procedure for the points (5,7) and (6,6):

$$\sqrt{(x-5)^2+(y-7)^2} = \sqrt{(x-6)^2+(y-6)^2}$$

Squaring each side gives

$$(x-5)^2+(y-7)^2 = (x-6)^2+(y-6)^2$$

or

$$x^2-10x+25+y^2-14y+49 =$$

$$x^2-12x+36+y^2-12y+36$$

Canceling and combining terms gives a second equation in x and y:

$$2x - 2y = -2$$

or

$$x - y = -1$$

Solving the equations simultaneously gives the coordinates of the intersection of the two perpendicular bisectors; this intersection is the center of the circle.

$$\begin{array}{rl} 3x - y = & 3 \\ \underline{x - y = } & \underline{-1} \text{ (Subtract)} \\ 2x \quad = & 4 \\ x \quad = & 2 \end{array}$$

Substitute the value $x = 2$ in one of the equations to find the value of y:

$$x - y = -1$$

$$2 - y = -1$$

$$-y = -3$$

$$y = 3$$

Thus, the center of the circle is the point (2,3).

The radius is the distance between the center (2,3) and one of the three given points. Using point (2,8), we obtain

$$r = \sqrt{(2 - 2)^2 + (8 - 3)^2} = \sqrt{25} = 5$$

The equation of this circle is

$$(x - 2)^2 + (y - 3)^2 = 25$$

as was found in the previous example.

If a circle is to be defined by three points, the points must be noncollinear. In some cases the three points are obviously noncollinear. Such is the case with the points (1,1), $(-2,2)$, and $(-1,-1)$, since these points cannot be connected by a straight line. However, in many cases you may find difficulty determining by inspection whether or not the points are collinear;

therefore, you need a method for determining this analytically. In the following example an attempt is made to find the circle determined by three points that are collinear.

EXAMPLE: Find the equation of the circle that passes through the points (1,1), (2,2), and (3,3).

SOLUTION: Substitute the given values of x and y in the general form of the equation of a circle to get three equations in three unknowns:

$$x^2 + y^2 + Bx + Cy + D = 0$$

For (1,1) $\quad 1 + 1 + B + C + D = 0$

$$B + C + D = -2 \qquad \text{Equation (1)}$$

For (2,2) $\quad 4 + 4 + 2B + 2C + D = 0$

$$2B + 2C + D = -8 \qquad (2)$$

For (3,3) $\quad 9 + 9 + 3B + 3C + D = 0$

$$3B + 3C + D = -18 \qquad (3)$$

To eliminate D, first subtract equation (1) from equation (2):

$$\begin{array}{rl} 2B + 2C + D = -8 & (2) \\ \underline{B + C + D = -2} & (-)\ (1) \\ B + C \phantom{{}+D} = -6 & (4) \end{array}$$

Next subtract equation (2) from equation (3):

$$\begin{array}{rl} 3B + 3C + D = -18 & (3) \\ \underline{2B + 2C + D = -8} & (-)\ (2) \\ B + C \phantom{{}+D} = -10 & (5) \end{array}$$

Then subtract equation (5) from equation (4) to eliminate one of the unknowns:

$$\begin{array}{rl} B + C = -6 & (4) \\ \underline{B + C = -10} & (-)\ (5) \\ 0 + 0 = 4 & \\ 0 = 4 & \end{array}$$

This solution is not valid, so no circle passes through the three given points. You should attempt to solve equations (4) and (5) by the substitution method. When the three given points are collinear, an inconsistent solution of some type will result.

If you try to solve the problem by eliminating both B and C at the same time (to find D), another type of inconsistent solution results. With the given coefficients you can easily eliminate both A and B at the same time. First, multiply equation (2) by 3 and equation (3) by -2 and add the resultant equations:

$$\begin{array}{rlr} 6B + 6C + 3D = & -24 & 3\ (\times)\ (2) \\ \underline{-6B - 6C - 2D =} & \underline{\ \ 36} & (+)\ -2\ (\times)\ (3) \\ D = & 12 & \end{array}$$

Then multiply equation (1) by -2 and add the resultant to equation (2):

$$\begin{array}{rlr} -2B - 2C - 2D = & 4 & -2\ (\times)\ (1) \\ \underline{2B + 2C + D =} & \underline{-8} & (+)\ (2) \\ -D = & -4 & \\ D = & 4 & \end{array}$$

This gives two values for D, which is inconsistent since each of the constants must have a unique value consistent with the given conditions. The three points are on the straight line $y = x$.

PRACTICE PROBLEMS:

In each of the following problems, find the equation of the circle that passes through the three given points:

1. (14,0), (12,4), and (3,7)

2. (10,3), (11,8), and (7,14)

3. (1,1), (0,0), and $(-1,-1)$

4. $(12,-5)$, $(-9,-12)$, and $(-4,3)$

ANSWERS:

1. $x^2 + y^2 - 10x + 4y = 56$

2. $x^2 + y^2 - 6x - 14y = 7$

3. No solution; the given points describe the straight line $y = x$.

4. $x^2 + y^2 - 2x + 14y = 75$

THE PARABOLA

A *parabola* is the locus of all points in a plane equidistant from a fixed point, called the *focus*, and a fixed line, called the *directrix*. In the parabola shown in figure 2-8, point V, which lies halfway between the focus and the directrix, is called the *vertex* of the parabola. In this figure and in many of the parabolas discussed in the first portion of this section, the vertex of the parabola falls at the origin; however, the vertex of the parabola, like the center of the circle, can fall at any point in the plane.

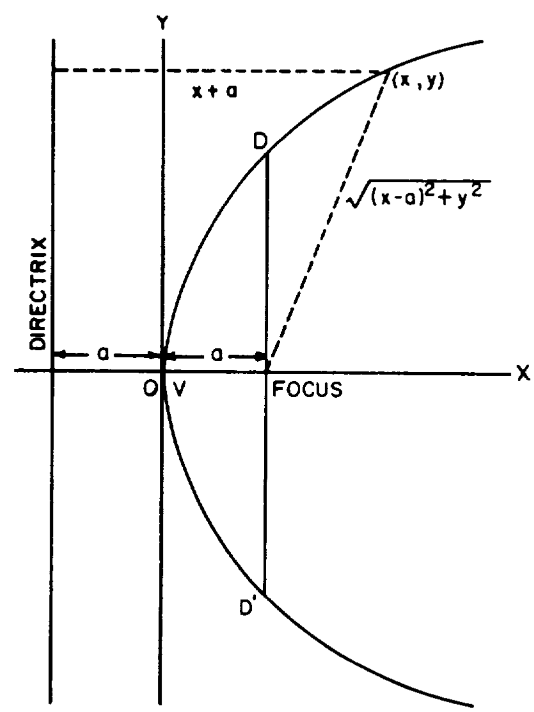

Figure 2-8.—The parabola.

The distance from the point (x,y) on the curve to the focus $(a,0)$ is

$$\sqrt{(x - a)^2 + y^2}$$

The distance from the point (x,y) to the directrix $x = -a$ is

$$x + a$$

Since by definition these two distances are equal, we may set them equal:

$$\sqrt{(x - a)^2 + y^2} = x + a$$

Squaring both sides, we have

$$(x - a)^2 + y^2 = (x + a)^2$$

Expanding, we have

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

Canceling and combining terms, we have an equation for the parabola:

$$y^2 = 4ax$$

For every positive value of x in the equation of the parabola, we have two values of y. But when x becomes negative, the values of y are imaginary. Thus, the curve must be entirely to the right of the Y axis when the equation is in this form. If the equation is

$$y^2 = -4ax$$

the curve lies entirely to the left of the Y axis.

If the form of the equation is

$$x^2 = 4ay$$

the curve opens upward and the focus is a point on the Y axis. For every positive value of y, you will have two values of x, and the curve will be entirely above the X axis. When the equation is in the form

$$x^2 = -4ay$$

the curve opens downward, is entirely below the X axis, and has as its focus a point on the negative Y axis. Parabolas that are representative of the four cases given here are shown in figure 2-9.

When x is equal to a in the equation

$$y^2 = 4ax$$

then

$$y^2 = 4a^2$$

and

$$y = 2a$$

This value of y is the height of the curve at the focus or the distance from the focus to point D in figure 2-8. The width of the curve at the focus, which is the distance from point D to point D' in

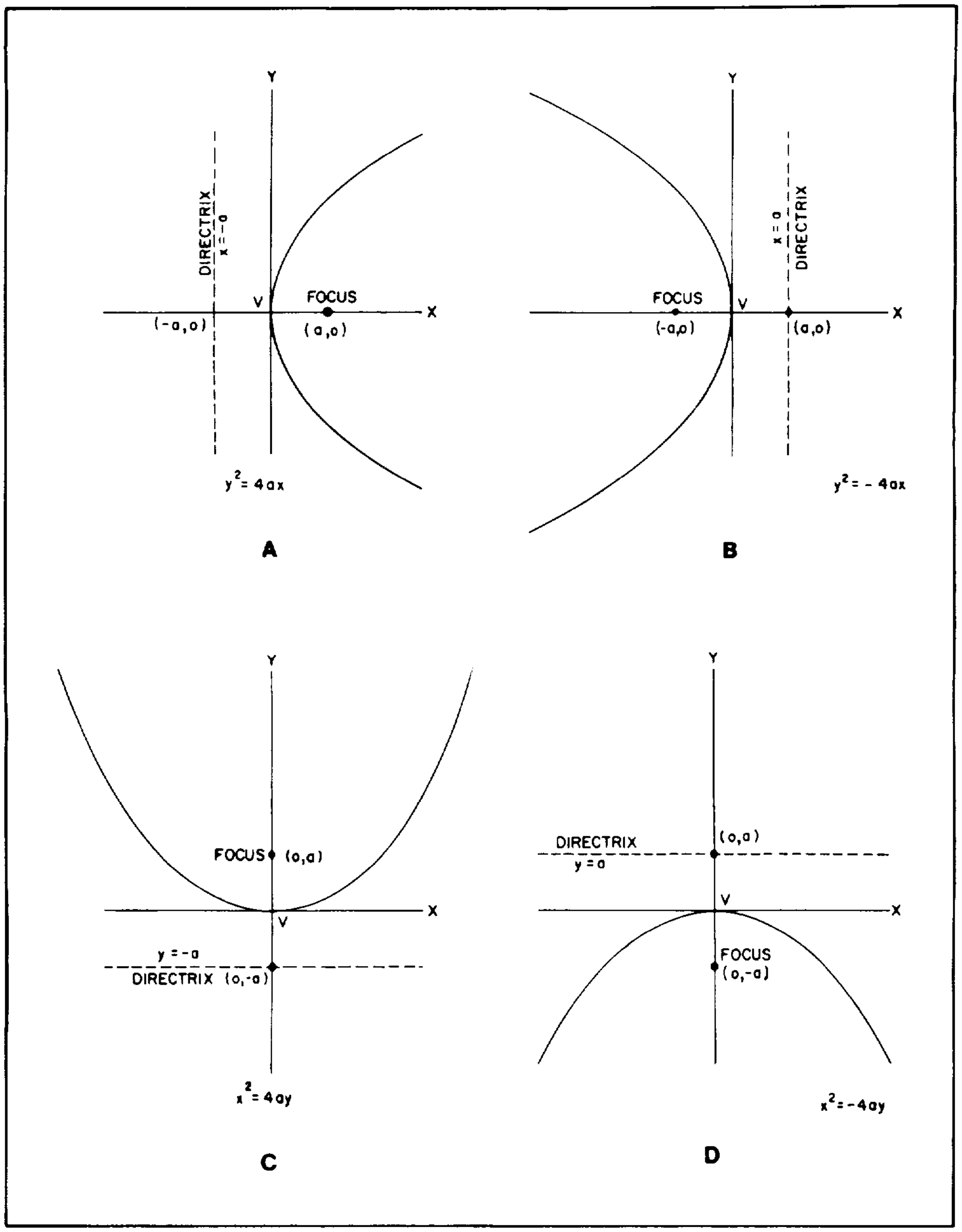

Figure 2-9.—Parabolas corresponding to four forms of the equation.

the figure, is equal to $4a$. This width is called the *focal chord*. The focal chord is one of the properties of a parabola used in the analysis of a parabola or in the sketching of a parabola.

EXAMPLE: Give the length of a; the length of the focal chord; and the equation of the parabola, which is the locus of all points equidistant from the point (3,0) and the line $x = -3$.

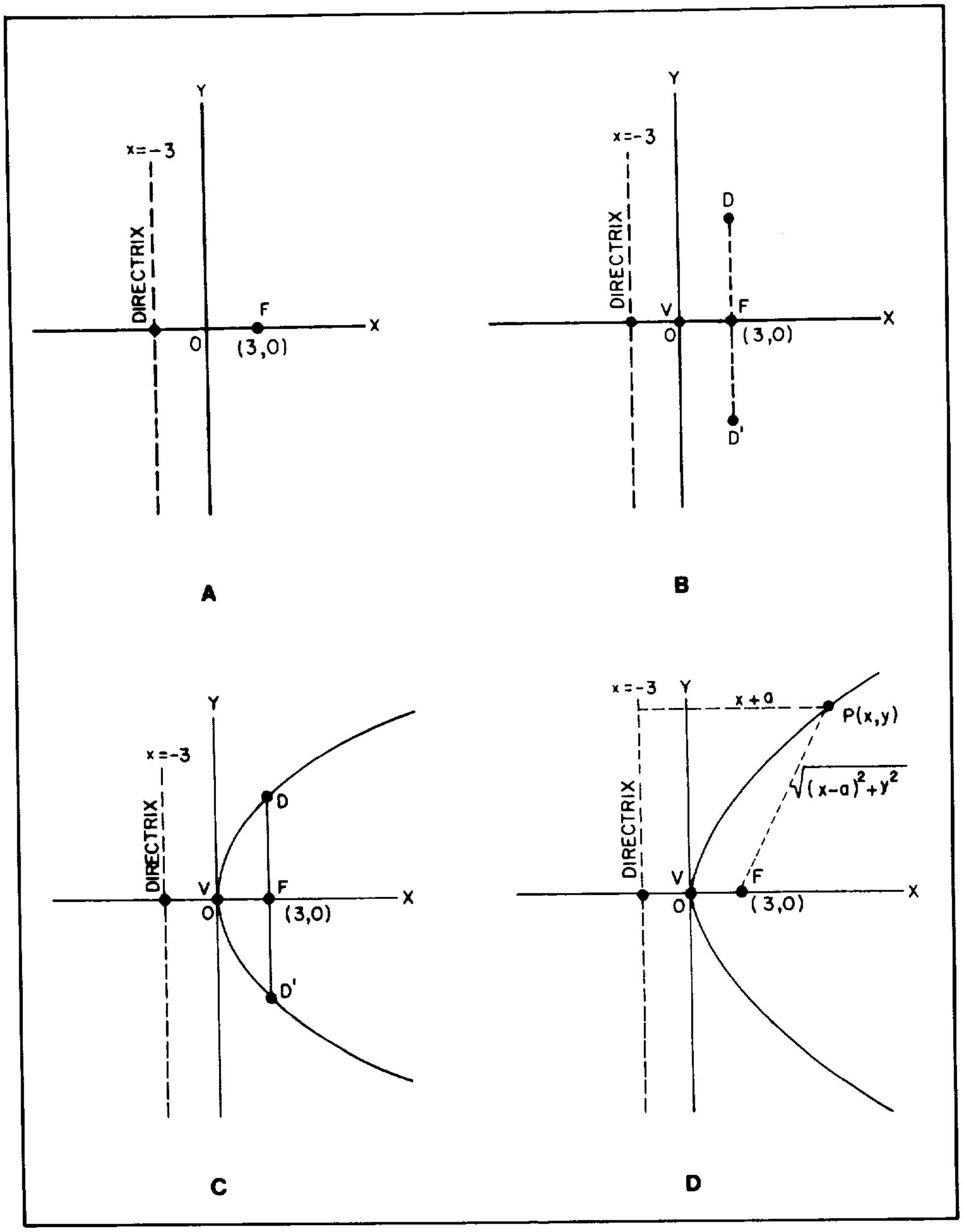

Figure 2-10.—Sketch of a parabola.

SOLUTION: First plot the given information on a coordinate system as shown in figure 2-10, view A. Figure 2-8 shows you that the point (3,0) corresponds to the position of the focus and that the line $x = -3$ is the directrix of the parabola. Figure 2-8 also shows you that the length of a is equal to one half the distance from the focus to the directrix or, in this problem, one half the distance from $x = -3$ to $x = 3$. Thus, the length of a is 3.

The second value required by the problem is the length of the focal chord. As stated previously, the focal chord length is equal to $4a$. The length of a was found to be 3, so the length of the focal chord is 12. Figure 2-8 shows that one extremity of the focal chord is a point on the curve $2a$ or 6 units above the focus, and the other extremity is a second point $2a$ or 6 units below the focus. Using this information and recalling that the vertex is one-half the distance from the focus to the directrix, plot three more points as shown in figure 2-10, view B.

Now a smooth curve through the vertex and the two points that are the extremities of the focal chord provide a sketch of the parabola in this problem. (See fig. 2-10, view C.)

To find the equation of the parabola, refer to figure 2-10, view D, and use the procedure used earlier. We know by definition that any point $P(x,y)$ on the parabola is equidistant from the focus and directrix. Thus, we equate these two distances:

$$\sqrt{(x - a)^2 + y^2} = x + a$$

However, we have found distance a to be equal to 3, so we substitute:

$$\sqrt{(x - 3)^2 + y^2} = x + 3$$

We square both sides:

$$(x - 3)^2 + y^2 = (x + 3)^2$$

Then we expand:

$$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9$$

We cancel and combine terms to obtain the equation of the parabola:

$$y^2 = 12x$$

If we check the consistency of our findings, we see that the form of the equation and the sketch agree with figure 2-9, view A. Also, the 12 in the right side of the equation corresponds to the $4a$ in the standard form, which is correct since we determined that the value of a was 3. Or, since the curve is entirely to the right of the Y axis, then we can apply the formula $y^2 = 4ax$ by substituting $a = 3$ to give

$$y^2 = 4(3)x$$

$$= 12x$$

NOTE: When the focus of a parabola lies on the Y axis, the equated distance equation is

$$\sqrt{(y - a)^2 + x^2} = y + a$$

PRACTICE PROBLEMS:

Give the equation; the length of a; and the length of the focal chord for the parabola, which is the locus of all points equidistant from the point and the line, given in the following problems:

1. The point $(-2,0)$ and the line $x = 2$
2. The point $(0,4)$ and the line $y = -4$
3. The point $(0,-1)$ and the line $y = 1$
4. The point $(1,0)$ and the line $x = -1$

ANSWERS:

1. $y^2 = -8x$, $a = 2$, $f.c. = 8$
2. $x^2 = 16y$, $a = 4$, $f.c. = 16$
3. $x^2 = -4y$, $a = 1$, $f.c. = 4$
4. $y^2 = 4x$, $a = 1$, $f.c. = 4$

Up to now, all of the parabolas we have dealt with have had a vertex at the origin and a corresponding equation in one of the four following forms:

1. $y^2 = 4ax$
2. $y^2 = -4ax$

3. $x^2 = 4ay$

4. $x^2 = -4ay$

We will now present four more forms of the equation of a parabola. Each one is a standardized parabola with its vertex at point $V(h,k)$. When the vertex is moved from the origin to the point $V(h,k)$, the x and y terms of the equation are replaced by $(x - h)$ and $(y - k)$. Then the standard equation for the parabola that opens to the right (fig. 2-9, view A) is

$$(y - k)^2 = 4a(x - h)$$

The four standard forms of the equations for parabolas with vertices at the point $V(h,k)$ are as follows:

1. $(y - k)^2 = 4a(x - h)$, corresponding to $y^2 = 4ax$, parabola opens to the right

2. $(y - k)^2 = -4a(x - h)$, corresponding to $y^2 = -4ax$, parabola opens to the left

3. $(x - h)^2 = 4a(y - k)$, corresponding to $x^2 = 4ay$, parabola opens upward

4. $(x - h)^2 = -4a(y - k)$, corresponding to $x^2 = -4ay$, parabola opens downward

The method for reducing an equation to one of these standard forms is similar to the method used for reducing the equation of a circle.

EXAMPLE: Reduce the equation

$$y^2 - 6y - 8x + 1 = 0$$

to standard form.

SOLUTION: Rearrange the equation so that the second-degree term and any first-degree terms of the same unknown are on the left side. Then group the unknown term appearing only in the first degree and all constants on the right:

$$y^2 - 6y = 8x - 1$$

Then complete the square in y:

$$y^2 - 6y + 9 = 8x - 1 + 9$$

$$(y - 3)^2 = 8x + 8$$

To get the equation in the form

$$(y - k)^2 = 4a(x - h)$$

factor an 8 out of the right side. Thus,

$$(y - 3)^2 = 8(x + 1)$$

is the equation of the parabola with its vertex at $(-1,3)$.

PRACTICE PROBLEMS:

Reduce the equations given in the following problems to standard form:

1. $x^2 + 4 = 4y$
2. $y^2 - 4x = 6y - 9$
3. $4x + 8y + y^2 + 20 = 0$
4. $4x - 12y + 40 + x^2 = 0$

ANSWERS:

1. $x^2 = 4(y - 1)$
2. $(y - 3)^2 = 4x$
3. $(y + 4)^2 = -4(x + 1)$
4. $(x + 2)^2 = 12(y - 3)$

THE ELLIPSE

An *ellipse* is a conic section with an eccentricity greater than 0 and less than 1.

Referring to figure 2-11, let

$$PO = a$$

$$FO = c$$

$$OM = d$$

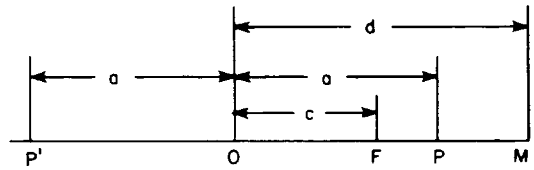

Figure 2-11.—Development of focus and directrix.

where F is the focus, O is the center, and P and P' are points on the ellipse. Then from the definition of eccentricity,

$$\frac{a - c}{d - a} = e \text{ or } a - c = ed - ea$$

and,

$$\frac{a + c}{d + a} = e \text{ or } a + c = ed + ea$$

Subtraction and addition of the two equations give

$$2c = 2ae \text{ or } c = ae \tag{2.4}$$

$$2a = 2de \text{ or } d = \frac{a}{e}$$

Place the center of the ellipse at the origin so that one focus lies at $(-ae,0)$ and one directrix is the line $x = -a/e$.

Figure 2-12 shows a point on the Y axis that satisfies the conditions for an ellipse. If

$$P''O = b$$

and

$$FO = c$$

then

$$P''F = b^2 + c^2$$

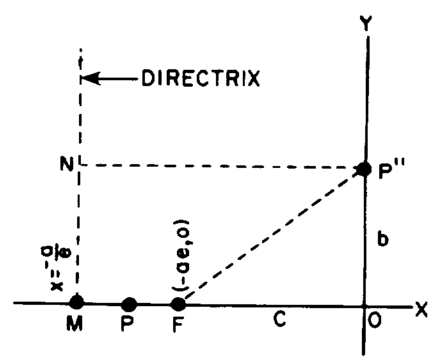

Figure 2-12.—Focus, directrix, and point P''.

By definition, e is the ratio of the distance of P'' from the focus and the directrix, so

$$e = \frac{P''F}{P''N}$$

or

$$e = \frac{\sqrt{b^2 + c^2}}{\frac{a}{e}}$$

Multiplying both sides by a/e gives

$$\sqrt{b^2 + c^2} = a$$

or

$$b^2 + c^2 = a^2$$

so

$$b^2 = a^2 - c^2 \tag{2.5}$$

Now combining equations (2.4) and (2.5) gives

$$b^2 = a^2 - a^2e^2$$

or

$$b^2 = a^2(1 - e^2) \tag{2.6}$$

Refer to figure 2-13. If the point (x,y) is on the ellipse, then the ratio of its distance from F to its distance from the directrix is e. The distance from (x,y) to the focus $(-ae,0)$ is

$$\sqrt{(x + ae)^2 + y^2}$$

and the distance from (x,y) to the directrix $x = -\frac{a}{e}$ is

$$x + \frac{a}{e}$$

The ratio of these two distances is equal to e, so

$$\frac{\sqrt{(x + ae)^2 + y^2}}{x + \frac{a}{e}} = e$$

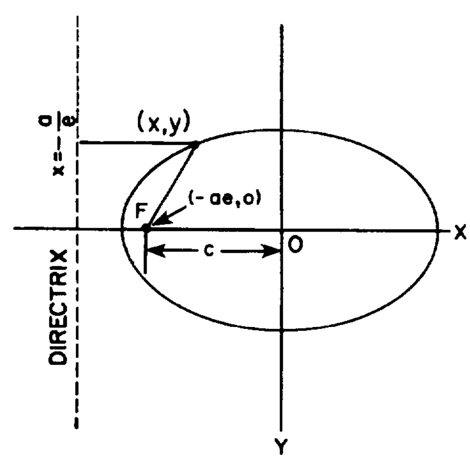

Figure 2-13.—The ellipse.

or

$$\sqrt{(x + ae)^2 + y^2} = e\left(x + \frac{a}{e}\right)$$
$$= ex + a$$

Squaring and expanding both sides gives

$$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2$$

Canceling like terms and transposing terms in x to the left-hand side of the equation gives

$$x^2 - e^2x^2 + y^2 = a^2 - a^2e^2$$

Removing a common factor gives

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2) \qquad (2.7)$$

Dividing both sides of equation (2.7) by the right-hand member gives

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

From equation (2.6) we obtain

$$b^2 = a^2(1 - e^2)$$

so that the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the equation of an ellipse in standard form. In figure 2-14, views A and B, a is the length of the semimajor axis and b is the length of the semiminor axis.

The curve is symmetrical with respect to the X and Y axes, so you can easily see that figure 2-14, view A, has another focus at $(ae,0)$ and a corresponding directrix, $x = a/e$. The curve also has vertices at $(\pm a,0)$.

The distance from the center through the focus to the curve is always designated a and is called the *semimajor axis*. This axis may be in either the x or y direction. When it is in the y direction, the directrix is a line denoted by the equation

$$y = k$$

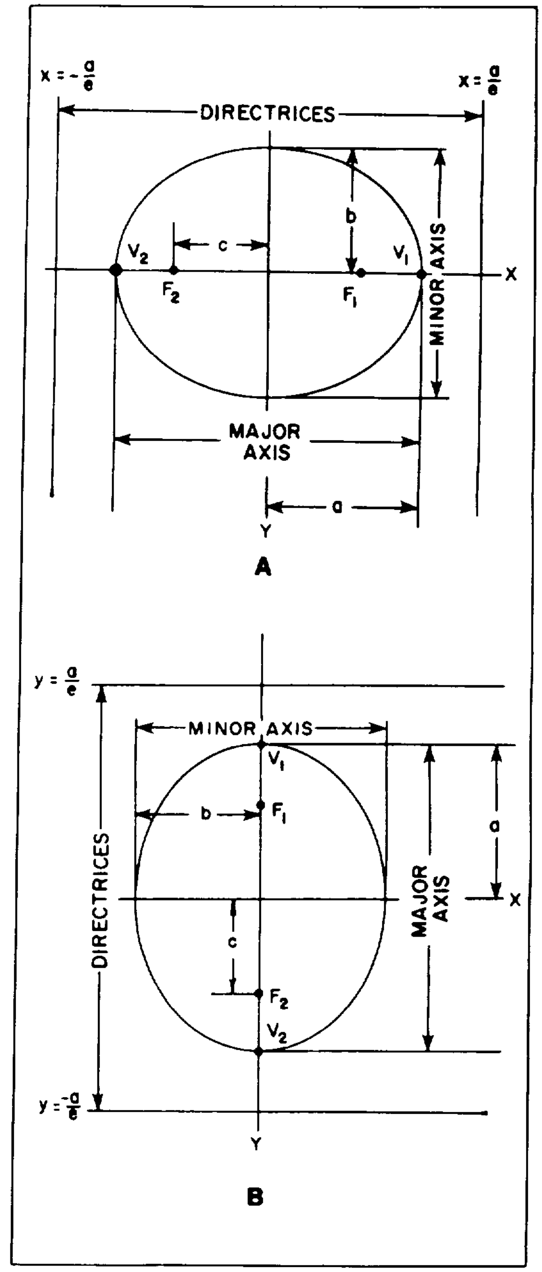

Figure 2-14.—Ellipse showing axes.

In the case we have studied, the directrix was denoted by the formula

$$x = k$$

where k is a constant equal to $\pm a/e$.

The perpendicular distance from the midpoint of the major axis to the curve is called the *semiminor axis* and is always signified by b.

The distance from the center of the ellipse to the focus is called c. In any ellipse the following relations are true for a, b, and c:

$$c = \sqrt{a^2 - b^2} \text{ or } c^2 = a^2 - b^2$$

$$b = \sqrt{a^2 - c^2} \text{ or } b^2 = a^2 - c^2$$

$$a = \sqrt{b^2 + c^2} \text{ or } a^2 = b^2 + c^2$$

Whenever the directrix is a line denoted by the equation $y = k$, the major axis is in the y direction and the equation of the ellipse is as follows:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Refer to figure 2-14, view B. This curve has foci at $(0, \pm c)$ and vertices at $(0, \pm a)$.

In an ellipse the position of the a^2 and b^2 terms indicates the orientation of the ellipse axis. As shown in figure 2-14, views A and B, value a is the semimajor or longer axis.

In the previous paragraphs formulas were given showing the relationship between a, b, and c. In the first portion of this discussion, a formula showing the relationship between a, c, and the eccentricity was given. These relationships are used to find the equation of an ellipse in the following example:

EXAMPLE: Find the equation of the ellipse with its center at the origin and having foci at $(\pm 2\sqrt{6}, 0)$ and an eccentricity equal to $\frac{2\sqrt{6}}{7}$.

SOLUTION: With the focal points on the X axis, the ellipse is oriented as in figure 2-14, view A, and the standard form of the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

With the center at the origin, the numerators of the fractions on the left are x^2 and y^2. The problem is to find the values of a and b.

The distance from the center to either of the foci is equal to c (fig. 2-14, view A), so in this problem

$$c = 2\sqrt{6}$$

from the given coordinates of the foci.

The values of a, c, and e (eccentricity) are related by

$$c = ae$$

or

$$a = \frac{c}{e}$$

From the known information, substitute the values of c and e,

$$a = \frac{2\sqrt{6}}{\frac{2\sqrt{6}}{7}}$$

$$a = 2\sqrt{6}\left(\frac{7}{2\sqrt{6}}\right)$$

and

$$a = 7$$

$$a^2 = 49$$

Then, using the formula

$$b = \sqrt{a^2 - c^2}$$

or

$$b^2 = a^2 - c^2$$

and substituting for a^2 and c^2,

$$b^2 = 49 - (2\sqrt{6})^2$$

$$b^2 = 49 - (4)(6)$$

$$b^2 = 49 - 24$$

gives the final required value of

$$b^2 = 25$$

Then the equation of the ellipse is

$$\frac{x^2}{49} + \frac{y^2}{25} = 1$$

PRACTICE PROBLEMS:

Find the equation of the ellipse with its center at the origin and for which the following properties are given:

1. Foci at $(\pm\sqrt{7},0)$ and an eccentricity of $\frac{\sqrt{7}}{4}$.
2. Length of the semiminor axis is 5 along the X axis and $e = \sqrt{11}/6$.
3. Vertices at $(\pm 4,0)$ and directrices $x = \pm 8\sqrt{3}/3$.
4. Foci at $(0, \pm 4)$ and vertices at $(0, \pm 5)$.

ANSWERS:

1. $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$ or $\frac{x^2}{16} + \frac{y^2}{9} = 1$
2. $\frac{x^2}{5^2} + \frac{y^2}{6^2} = 1$ or $\frac{x^2}{25} + \frac{y^2}{36} = 1$
3. $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$ or $\frac{x^2}{16} + \frac{y^2}{4} = 1$
4. $\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$ or $\frac{x^2}{9} + \frac{y^2}{25} = 1$

An *ellipse* may be defined as the locus of all points in a plane, the sum of whose distances from two fixed points, called the foci, is a constant equal to $2a$. This is shown as follows:

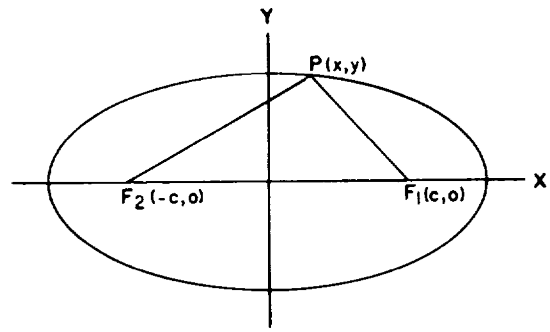

Figure 2-15.—Ellipse, center at origin.

Let the foci be F_1 and F_2 at $(\pm c,0)$, as shown in figure 2-15. Using the standard form of an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

solve for y^2:

$$y^2 = \frac{(a^2 - c^2)(a^2 - x^2)}{a^2}$$

Referring to figure 2-15, we see that

$$F_1P = \sqrt{(x - c)^2 + y^2}$$

and

$$F_2P = \sqrt{(x + c)^2 + y^2}$$

Substitute y^2 into both equations above and simplify

$$F_1P = \sqrt{(x - c)^2 + \frac{(a^2 - c^2)(a^2 - x^2)}{a^2}}$$

$$= a - \frac{cx}{a}$$

and

$$F_2P = \sqrt{(x + c)^2 + \frac{(a^2 - c^2)(a^2 - x^2)}{a^2}}$$

$$= a + \frac{cx}{a}$$

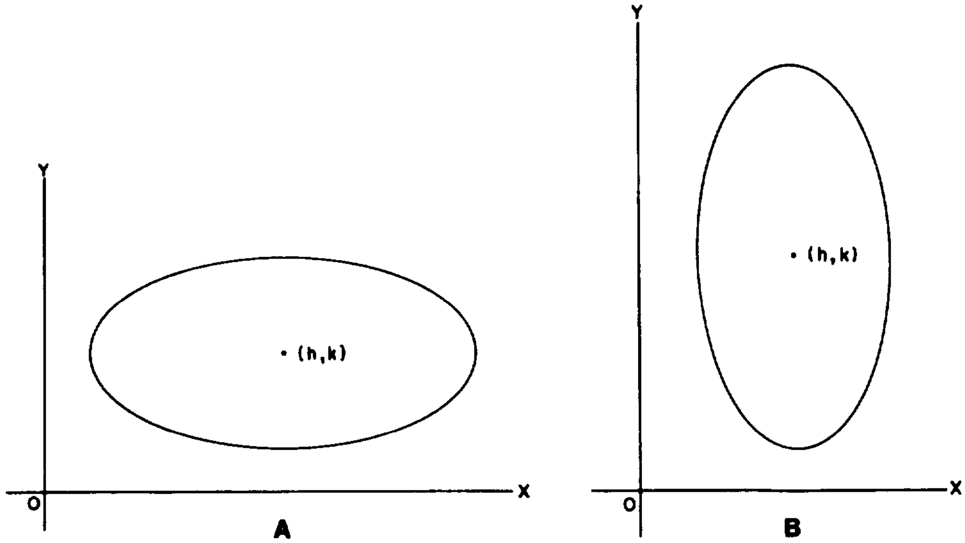

Figure 2-16.—Ellipse, center at (h,k).

so

$$F_1P + F_2P = a - \frac{cx}{a} + a + \frac{cx}{a}$$

$$= 2a$$

Whenever the center of the ellipse is at some point other than (0,0), such as the point (h,k) in figure 2-16, views A and B, the equation of the ellipse must be modified to the following standard forms:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad (2.8)$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \qquad (2.9)$$

Subtracting h from the value of x reduces the value of the term $(x - h)$ to the value x would have if the center were at the origin. The term $(y - k)$ is identical in value to the value of y if the center were at the origin.

Whenever we have an equation in the general form, such as

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where the capital letters refer to independent constants and A and C have the same sign, we can reduce the equation to the standard form for an ellipse. Completing the square in both x and y and performing a few simple algebraic transformations will change the form to that of equations (2.8) and (2.9).

Theorem: *An equation of the second degree, in which the xy term does not exist and the coefficients of* x^2 *and* y^2 *are different but have the same sign, represents an ellipse with axes parallel to the coordinate axes.*

EXAMPLE: Reduce the equation

$$4x^2 + 9y^2 - 40x - 54y + 145 = 0$$

to an ellipse in standard form.

SOLUTION: Collect terms in x and y and remove the common factors of these terms:

$$4x^2 - 40x + 9y^2 - 54y + 145 = 0$$

$$4(x^2 - 10x) + 9(y^2 - 6y) + 145 = 0$$

Transpose the constant terms and complete the square in both x and y. When factored terms are involved in completing the square, as in this example, an error is frequently made. The factored value operates on the term added inside the parentheses as well as the original terms. Therefore, the values added to the right side of the equation are the products of the factored values and the terms added to complete the square:

$$\begin{aligned} 4(x^2 - 10x + 25) + 9(y^2 - 6y + 9) &= -145 + 4(25) + 9(9) \\ &= -145 + 100 + 81 \\ &= 36 \end{aligned}$$

$$4(x - 5)^2 + 9(y - 3)^2 = 36$$

Divide both sides by the right-hand (constant) term. This reduces the right member to 1 as required by the standard form:

$$\frac{4(x-5)^2}{36} + \frac{9(y-3)^2}{36} = 1$$

$$\frac{(x-5)^2}{9} + \frac{(y-3)^2}{4} = 1$$

This reduces to the standard form

$$\frac{(x-5)^2}{(3)^2}+\frac{(y-3)^2}{(2)^2}=1$$

corresponding to equation (2.8). This equation represents an ellipse with the center at (5,3); its semimajor axis, a, equal to 3; and its semiminor axis, b, equal to 2.

EXAMPLE: Reduce the equation

$$3x^2+y^2+20x+32=0$$

to an ellipse in standard form.

SOLUTION: First, collect terms in x and y. As in the previous example, the coefficients of x^2 and y^2 must be reduced to 1 to complete the square in both x and y. Thus the coefficient of the x^2 term is divided out of the two terms containing x, as follows:

$$3x^2+20x+y^2+32=0$$

$$3\left(x^2+\frac{20x}{3}\right)+y^2=-32$$

Complete the square in x, noting that a product is added to the right side:

$$3\left(x^2+\frac{20x}{3}+\frac{100}{9}\right)+y^2=-32+3\left(\frac{100}{9}\right)$$

$$3\left(x+\frac{10}{3}\right)^2+y^2=\frac{-288+300}{9}$$

$$3\left(x+\frac{10}{3}\right)^2+y^2=\frac{12}{9}$$

$$3\left(x+\frac{10}{3}\right)^2+y^2=\frac{4}{3}$$

Divide both sides by the right-hand term:

$$\frac{3\left(x+\frac{10}{3}\right)^2}{\frac{4}{3}}+\frac{y^2}{\frac{4}{3}}=1$$

$$\frac{\left(x+\frac{10}{3}\right)^2}{\left(\frac{4}{9}\right)}+\frac{y^2}{\left(\frac{4}{3}\right)}=1$$

This equation reduces to the standard form,

$$\frac{\left(x+\frac{10}{3}\right)^2}{\left(\frac{2}{3}\right)^2}+\frac{y^2}{\left(\frac{2}{\sqrt{3}}\right)^2}=1$$

$$\frac{\left(x+\frac{10}{3}\right)^2}{\left(\frac{2}{3}\right)^2}+\frac{y^2}{\left(\frac{2\sqrt{3}}{3}\right)^2}=1$$

corresponding to equation (2.9), and represents an ellipse with the center at $\left(-\frac{10}{3},0\right)$.

PRACTICE PROBLEMS:

Express the following equations as an ellipse in standard form:

1. $5x^2 - 110x + 4y^2 + 425 = 0$

2. $x^2 - 14x + 36y^2 - 216y + 337 = 0$

3. $9x^2 - 54x + 4y^2 + 16y + 61 = 0$

4. $3x^2 - 14x + 4y^2 + 11 = 0$

ANSWERS:

1. $\frac{(x-11)^2}{(6)^2}+\frac{y^2}{(3\sqrt{5})^2}=1$

2. $\frac{(x-7)^2}{(6)^2}+\frac{(y-3)^2}{(1)^2}=1$

3. $\dfrac{(x-3)^2}{(2)^2} + \dfrac{(y+2)^2}{(3)^2} = 1$

4. $\dfrac{\left(x - \frac{7}{3}\right)^2}{\left(\frac{4}{3}\right)^2} + \dfrac{y^2}{\left(\frac{2\sqrt{3}}{3}\right)^2} = 1$

THE HYPERBOLA

A *hyperbola* is a conic section with an eccentricity greater than 1.

The formulas

$$c = ae$$

and

$$d = \frac{a}{e}$$

developed in the section concerning the ellipse were derived so that they are true for any value of eccentricity. Thus, they are true for the hyperbola as well as for an ellipse. Since e is greater than 1 for a hyperbola, then

$$c = ae \text{ and } c > a$$

$$d = \frac{a}{e} \text{ and } d < a$$

Therefore $c > a > d$.

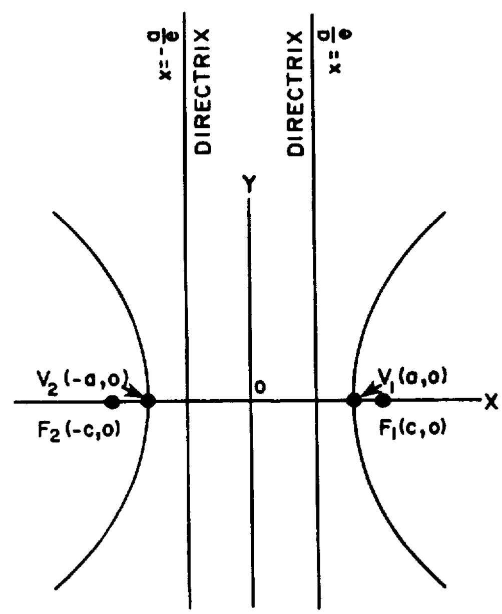

Figure 2-17.—The hyperbola.

According to this analysis, if the center of symmetry of a hyperbola is the origin, then the foci lies farther from the origin than the directrices. An inspection of figure 2-17 shows that the curve never crosses the Y axis. Thus the solution for the value of b, the semiminor axis of the ellipse, yields no real value for b. In other words, b is an imaginary number. This can easily be seen from the equation

$$b = \sqrt{a^2 - c^2}$$

since $c > a$ for a hyperbola.

However, we can square both sides of the the above equation, and since the square of an imaginary number is a negative real number we write

$$-b^2 = a^2 - c^2$$

or

$$b^2 = c^2 - a^2$$

and, since $c = ae$,

$$b^2 = a^2e^2 - a^2 = a^2(e^2 - 1)$$

Now we can use this equation to obtain the equation of a hyperbola from the following equation, which was developed in the section on the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

and since

$$a^2(1 - e^2) = -a^2(e^2 - 1) = -b^2$$

we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is a standard form for the equation of a hyperbola with its center, O, at the origin. The solution of this equation for y gives

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

which shows that y is imaginary only when $x^2 < a^2$. The curve, therefore, lies entirely beyond the two lines $x = \pm a$ and crosses the X axis at $V_1(a,0)$ and $V_2(-a,0)$, the vertices of the hyperbola.

The two straight lines

$$bx + ay = 0 \text{ and } bx - ay = 0 \qquad (2.10)$$

can be used to illustrate an interesting property of a hyperbola. The distance from the line $bx - ay = 0$ to the point (x_1,y_1) on the curve is given by

$$d = \frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}} \qquad (2.11)$$

Since (x_1,y_1) is on the curve, its coordinates satisfy the equation

$$b^2x_1^2 - a^2y_1^2 = a^2b^2$$

which may be written

$$(bx_1 - ay_1)(bx_1 + ay_1) = a^2b^2$$

or

$$bx_1 - ay_1 = \frac{a^2b^2}{bx_1 + ay_1}$$

Now substituting this value into equation (2.11) gives us

$$d = \frac{a^2b^2}{\sqrt{a^2 + b^2}}\left(\frac{1}{bx_1 + ay_1}\right)$$

As the point (x_1,y_1) is chosen farther and farther from the center of the hyperbola, the absolute values for x_1 and y_1 will increase and the distance, d, will approach zero. A similar result can easily be derived for the line $bx + ay = 0$.

The lines of equation (2.10), which are usually written

$$y = -\frac{b}{a}x \text{ and } y = +\frac{b}{a}x$$

are called the *asymptotes* of the hyperbola. They are very important in tracing a curve and studying its properties. The

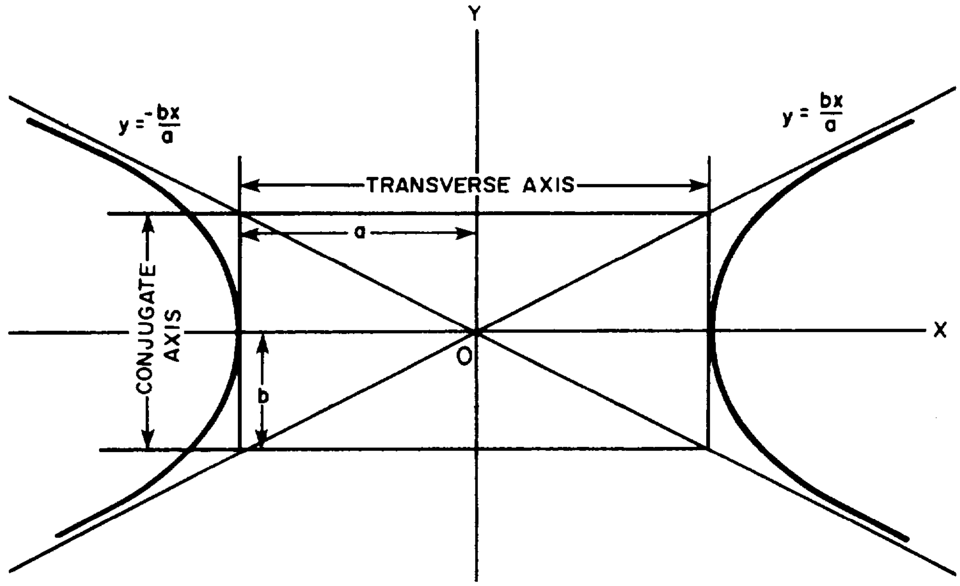

Figure 2-18.—Using asymptotes to sketch a hyperbola.

asymptotes of a hyperbola, figure 2-18, are the diagonals of the rectangle whose center is the center of the curve and whose sides are parallel and equal to the axes of the curve. The *focal chord* of a hyperbola is equal to $\frac{2b^2}{a}$.

Another definition of a *hyperbola* is the locus of all points in a plane such that the difference of their distances from two fixed points is constant. The fixed points are the *foci*, and the constant difference is $2a$.

The nomenclature of the hyperbola is slightly different from that of an ellipse. The *transverse axis* is of length $2a$ and is the distance between the intersections (vertices) of the hyperbola with its focal axis. The *conjugate axis* is of length $2b$ and is perpendicular to the transverse axis.

Whenever the foci are on the Y axis and the directrices are lines of the form $y = \pm k$, where k is a constant, the equation of the hyperbola will read

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

This equation represents a hyperbola with its transverse axis on the Y axis. Its asymptotes are the lines $by - ax = 0$ and $by + ax = 0$ or

$$x = \frac{b}{a}y \text{ and } x = -\frac{b}{a}y$$

The properties of the hyperbola most often used in analysis of the curve are the foci, directrices, length of the focal chord, and the equations of the asymptotes.

Figure 2-17 shows that the foci are given by the points $F_1(c,0)$ and $F_2(-c,0)$ when the equation of the hyperbola is in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If the equation were

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

the foci would be the points $(0,c)$ and $(0,-c)$. The value of c is either determined from the formula

$$c^2 = a^2 + b^2$$

or the formula

$$c = ae$$

Figure 2-17 also shows that the directrices are the lines $x = \pm\frac{a}{e}$ or, in the case where the hyperbolas open upward and downward, $y = \pm\frac{a}{e}$. This was also given earlier in this discussion as $d = \frac{a}{e}$.

The equations of the asymptotes were given earlier as

$$bx + ay = 0 \text{ and } bx - ay = 0$$

or

$$y = -\frac{b}{a}x \text{ and } y = +\frac{b}{a}x$$

The earlier reference also pointed out that the length of the focal chord is equal to $\frac{2b^2}{a}$.

Note that you have no restriction of $a > b$ for the hyperbola as you have for the ellipse. Instead, the direction in which the hyperbola opens corresponds to the transverse axis on which the foci and vertices lie.

The properties of a hyperbola can be determined from the equation of a hyperbola or the equation can be written given certain properties, as shown in the following examples. In these examples and in the practice problems immediately following, all of the hyperbolas considered have their centers at the origin.

EXAMPLE: Find the equation of the hyperbola with an eccentricity of 3/2, directrices $x = \pm 4/3$, and foci at $(\pm 3,0)$.

SOLUTION: The foci lie on the X axis at the points (3,0) and $(-3,0)$, so the equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This fact is also shown by the equation of the directrices.

Since we have determined the form of the equation and since the center of the curve in this section is restricted to the origin, the problem is reduced to finding the values of a^2 and b^2.

First, the foci are given as $(\pm 3,0)$; and since the foci are also the points $(\pm c,0)$, then

$$c = 3$$

The eccentricity is given and the value of a^2 can be determined from the formula

$$c = ae$$

$$a = \frac{c}{e}$$

$$a = \frac{3}{\frac{3}{2}}$$

$$a = \frac{6}{3}$$

$$a = 2$$

$$a^2 = 4$$

The relationship of a, b, and c for the hyperbola is

$$b^2 = c^2 - a^2$$

and

$$b^2 = (3)^2 - (2)^2$$

$$b^2 = 9 - 4$$

$$b^2 = 5$$

When these values are substituted in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

the equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

results and is the equation of the hyperbola.

The equation could also be found by the use of other relationships using the given information.

The directrices are given as

$$x = \pm\frac{4}{3}$$

and, since

$$d = \frac{a}{e}$$

or

$$a = de$$

substituting the values given for d and e results in

$$a = \frac{4}{3}\left(\frac{3}{2}\right)$$

therefore,

$$a = 2$$

and

$$a^2 = 4$$

While the value of c can be determined by the given information in this problem, it could also be computed since

$$c = ae$$

and a has been found to equal 2 and e is given as $\frac{3}{2}$; therefore,

$$c = 2\left(\frac{3}{2}\right)$$

$$= 3$$

With values for a and c computed, the value of b is found as before and the equation can be written.

EXAMPLE: Find the foci, directrices, eccentricity, length of the focal chord, and equations of the asymptotes of the hyperbola described by the equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

SOLUTION: This equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and the values for a and b are determined by inspection to be

$$a^2 = 9$$

$$a = 3$$

and

$$b^2 = 16$$

$$b = 4$$

With a and b known, we find c by using the formula

$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$

$$c = \sqrt{9 + 16}$$

$$c = \sqrt{25}$$

$$c = 5$$

From the form of the equation, we know that the foci are at the points

$$F_1(c,0)$$

and

$$F_2(-c,0)$$

so the foci = $(\pm 5,0)$.

The eccentricity is found by the formula

$$e = \frac{c}{a}$$

$$e = \frac{5}{3}$$

Figure 2-17 shows that with the center at the origin, c and a will have the same sign.

The directrix is found by the formula

$$d = \frac{a}{e}$$

or, since this equation will have directrices parallel to the Y axis, by the formula

$$x = \pm \frac{a}{e}$$

Then

$$x = \pm \frac{3}{\frac{5}{3}}$$

$$x = \pm 3 \left(\frac{3}{5}\right)$$

So the directrices are the lines

$$x = \pm\frac{9}{5}$$

The focal chord (*f.c.*) is found by

$$f.c. = \frac{2b^2}{\text{a}}$$

$$f.c. = \frac{2(16)}{3}$$

$$f.c. = \frac{32}{3}$$

Finally, the equations of the asymptotes are the equations of the two straight lines:

$$bx + ay = 0$$

and

$$bx - ay = 0$$

In this problem, substituting the values of a and b in each equation gives

$$4x + 3y = 0$$

and

$$4x - 3y = 0$$

or

$$4x \pm 3y = 0$$

The equations of the lines asymptotic to the curve can also be written in the form

$$y = \frac{b}{a}x$$

and

$$y = -\frac{b}{a}x$$

In this form the lines are

$$y = \frac{4}{3}x$$

and

$$y = -\frac{4}{3}x$$

or

$$y = \pm\frac{4}{3}x$$

If we think of this equation as a form of the slope-intercept formula

$$y = mx + b$$

from chapter 1, the lines would have slopes of $\pm\frac{b}{a}$ and each would have its y intercept at the origin as shown in figure 2-18.

PRACTICE PROBLEMS:

1. Find the equation of the hyperbola with an eccentricity of $\sqrt{2}$, directrices $y = \pm\frac{\sqrt{2}}{2}$, and foci at $(0, \pm\sqrt{2})$.

2. Find the equation of the hyperbola with an eccentricity of 5/3, foci at $(\pm 5,0)$, and directrices $x = \pm 9/5$.

Find the foci, directrices, eccentricity, equations of the asymptotes, and length of the focal chord of the hyperbolas given in problems 3 and 4.

3. $\frac{x^2}{9} - \frac{y^2}{9} = 1$

4. $\frac{y^2}{9} - \frac{x^2}{4} = 1$

ANSWERS:

1. $y^2 - x^2 = 1$

2. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

3. foci $= (\pm 3\sqrt{2},0)$; directrices $x = \frac{\pm 3}{\sqrt{2}}$;
   eccentricity $= \sqrt{2}$; $f.c. = 6$; asymptotes $y = \pm x$

4. foci $= (0, \pm\sqrt{13})$; directrices $y = \frac{\pm 9}{\sqrt{13}}$;
   eccentricity $= \frac{\sqrt{13}}{3}$; $f.c. = \frac{8}{3}$; asymptotes $x = \pm\frac{2}{3}y$

The hyperbola can be represented by an equation in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where the capital letters refer to independent constants and A and C have different signs. These equations can be reduced to standard form in the same manner in which similar equations for the ellipse were reduced to standard form. The standard forms with the center at (h,k) are given by the equations

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

and

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

POLAR COORDINATES

So far we have located a point in a plane by giving the distances of the point from two perpendicular lines. We can define the location of a point equally well by noting its distance and bearing. This method is commonly used aboard ship to show the position of another ship or target. Thus, 3 miles at 35° locates the position of a ship relative to the course of the ship making the reading. We can use this method to develop curves and bring out their properties. Assume a fixed direction OX and a fixed point O on the line in figure 2-19. The position of any point, P, is fully determined if we know the directed distance from O to P and the angle that the line OP makes with reference line OX. The line OP is called the *radius vector* and the angle POX is called the *polar angle*. The radius vector is denoted by ϱ, while θ denotes the polar angle.

Point O is the *pole* or *origin*. As in conventional trigonometry, *the polar angle is positive when measured counterclockwise and negative when measured clockwise*. However, unlike the convention established in trigonometry, *the radius vector for polar coordinates is positive only when it is laid off on the terminal side of the angle. When the radius vector is laid off on the terminal side of the ray produced beyond the pole (the given angle plus 180°), a negative value is assigned the radius vector*. For this reason, more than one equation may be used in polar coordinates to describe a given locus. It is sufficient that you remember that the radius vector can be negative. In this course, however, the radius vector, ϱ, will always be positive.

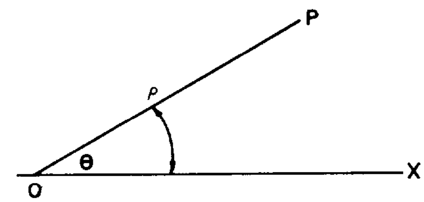

Figure 2-19.—Defining the polar coordinates.

TRANSFORMATION FROM CARTESIAN TO POLAR COORDINATES

At times you will find working with the equation of a curve in polar coordinates will be easier than working in Cartesian coordinates. Therefore, you need to know how to change from one system to the other. Sometimes both forms are useful, for some properties of the curve may be more apparent from one form of the equation.

We can make transformations by applying the following equations, which can be derived from figure 2-20:

$$x = \varrho \cos \theta \qquad (2.12)$$

$$y = \varrho \sin \theta$$

$$\varrho^2 = x^2 + y^2 \qquad (2.13)$$

$$\tan \theta = \frac{y}{x}$$

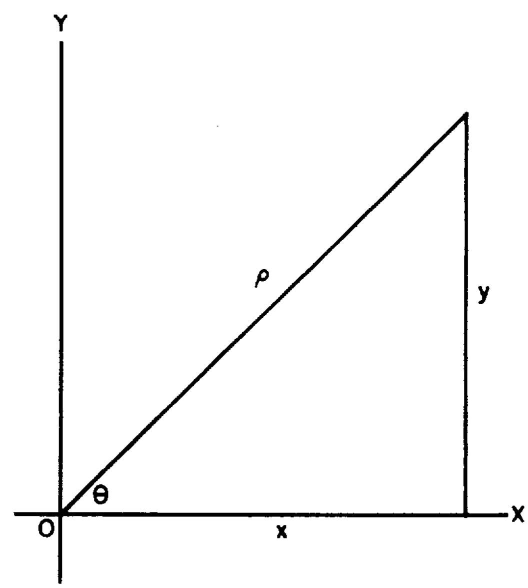

Figure 2-20.—Cartesian and polar relationship.

EXAMPLE: Change the equation

$$y = x^2$$

from rectangular to polar coordinates.

SOLUTION: Substitute $\varrho \cos \theta$ for x and $\varrho \sin \theta$ for y so that we have

$$\varrho \sin \theta = \varrho^2 \cos^2\theta$$

$$\sin \theta = \varrho \cos^2\theta$$

or

$$\varrho = \frac{\sin \theta}{\cos^2\theta}$$

$$\varrho = \tan \theta \sec \theta$$

EXAMPLE: Express the equation of the following circle with its center at $(a,0)$ and with radius a, as shown in figure 2-21, in polar coordinates:

$$(x - a)^2 + y^2 = a^2$$

SOLUTION: First, expanding this equation gives us

$$x^2 - 2ax + a^2 + y^2 = a^2$$

Rearranging terms, we have

$$x^2 + y^2 = 2ax$$

The use of equation (2.13) gives us

$$\varrho^2 = 2ax$$

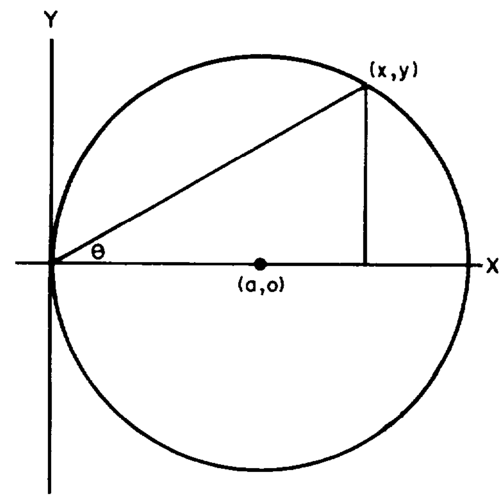

Figure 2-21.—Circle with center $(a,0)$.

and applying the value of x given by equation (2.12), results in

$$\varrho^2 = 2a\varrho \cos \theta$$

Dividing both sides by ϱ, we have the equation of a circle with its center at $(a,0)$ and radius a in polar coordinates

$$\varrho = 2a \cos \theta$$

TRANSFORMATION FROM POLAR TO CARTESIAN COORDINATES

To transform to an equation in Cartesian or rectangular coordinates from an equation in polar coordinates, use the following equations, which can be derived from figure 2-22:

$$\varrho = \sqrt{x^2 + y^2} \qquad (2.14)$$

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\tan \theta = \frac{y}{x} \qquad (2.15)$$

$$\sec \theta = \frac{\sqrt{x^2 + y^2}}{x} \qquad (2.16)$$

$$\csc \theta = \frac{\sqrt{x^2 + y^2}}{y}$$

$$\cot \theta = \frac{x}{y}$$

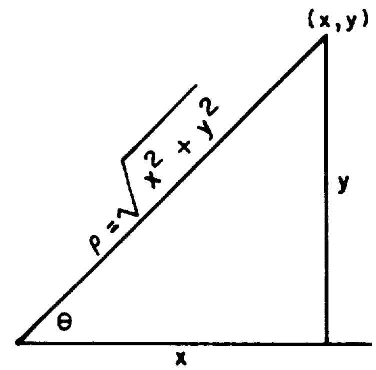

Figure 2-22.—Polar to cartesian relationship.

EXAMPLE: Change the equation

$$\varrho = \sec \theta \tan \theta$$

to an equation in rectangular coordinates.

SOLUTION: Applying relations (2.14), (2.15), and (2.16) to the above equation gives

$$\sqrt{x^2 + y^2} = \frac{\sqrt{x^2 + y^2}}{x}\left(\frac{y}{x}\right)$$

Dividing both sides by $\sqrt{x^2 + y^2}$, we obtain

$$1 = \left(\frac{y}{x^2}\right)$$

or

$$y = x^2$$

which is the equation we set out to find.

EXAMPLE: Change the following equation to an equation in rectangular coordinates:

$$\varrho = \frac{3}{\sin\theta - 3\cos\theta}$$

SOLUTION: Written without a denominator, the polar equation is

$$\varrho \sin\theta - 3\varrho \cos\theta = 3$$

Using the transformations

$$\varrho \sin\theta = y$$

$$\varrho \cos\theta = x$$

we have

$$y - 3x = 3$$

as the equation in rectangular coordinates.

PRACTICE PROBLEMS:

Change the equations in problems 1 through 4 to equations having polar coordinates.

1. $x^2 + y^2 = 4$
2. $\left(x^2 + y^2\right) = 9\left(\frac{y}{x}\right)^2$
3. $3y - 7x = 10$
4. $y = 2x - 3$

Change the equations in the following problems to equations having Cartesian coordinates.

5. $\varrho = 4 \sin\theta$
6. $\varrho = \sin\theta + \cos\theta$
7. $\varrho = a^2$

ANSWERS:

1. $\varrho = \pm 2$

2. $\varrho = 3 \tan \theta$

3. $\varrho = \dfrac{10}{3 \sin \theta - 7 \cos \theta}$

4. $\varrho = \dfrac{-3}{\sin \theta - 2 \cos \theta}$

5. $x^2 + y^2 - 4y = 0$

6. $x^2 + y^2 = y + x$

7. $x^2 + y^2 = a^4$

SUMMARY

The following are the major topics covered in this chapter:

1. **Conic section:** A *conic section* is the locus of all points in a plane whose distance from a fixed point is a constant ratio to its distance from a fixed line. The fixed point is the *focus,* and the fixed line is the *directrix*. The ratio referred to is called the *eccentricity*.

2. **Eccentricity:**

   If $0 < e < 1$, then the curve is an *ellipse.*

   If $e > 1$, then the curve is a *hyperbola.*

   If $e = 1$, then the curve is a *parabola.*

   If $e = 0$, then the curve is a *circle.*

3. **Locus of an equation:** The *locus* of an equation is a curve containing those points, and only those points, whose coordinates satisfy the equation.

4. **Circle:** A *circle* is the locus of all points, in a plane that is always a fixed distance, called the *radius*, from a fixed point, called the *center.*

   Theorem: *An equation of the second degree, in which the coefficients of the x^2 and y^2 terms are equal and the xy term does not exist represents a circle.*

5. **Standard equation of a circle:**

   $$(x - h)^2 + (y - k)^2 = r^2$$

   where (x,y) is a point on the circle, (h,k) is the center, and r is the radius of the circle.

6. **General equation of a circle:**

   $$x^2 + y^2 + Bx + Cy + D = 0$$

   where B, C, and D are constants.

7. **Circle defined by three points:** A circle may be defined by three noncollinear points; that is, by three points not lying on

a straight line. Only one circle is possible through any three noncollinear points. To find the equation of the circle determined by three points, substitute the x and y values of each of the given points into the general equation to form three equations with B, C, and D as the unknowns. These equations are then solved simultaneously to find the values of B, C, and D in the equation that satisfies the three given conditions.

8. **Parabola:** A *parabola* is the locus of all points in a plane equidistant from a fixed point, called the *focus*, and a fixed line, called the *directrix*.

   The point which lies halfway between the focus and the directrix is called the *vertex*.

   The *focal chord* is equal to $4a$, where a is the distance from the vertex to the focus.

   A parabola with its vertex at the origin and opening to the right has its focus at $(a,0)$ and its directrix at $x = -a$; its corresponding equation is $y^2 = 4ax$.

   A parabola with its vertex at the origin and opening to the left has its focus at $(-a,0)$ and its directrix at $x = a$; its corresponding equation is $y^2 = -4ax$.

   A parabola with its vertex at the origin and opening upward has its focus at $(0,a)$ and its directrix at $y = -a$; its corresponding equation is $x^2 = 4ay$.

   A parabola with its vertex at the origin and opening downward has its focus at $(0,-a)$ and its directrix at $y = a$; its corresponding equation is $x^2 = -4ay$.

9. **Standard equations for parabolas:**

   1. $(y - k)^2 = 4a(x - h)$ (parabola opening to the right)
   2. $(y - k)^2 = -4a(x - h)$ (parabola opening to the left)
   3. $(x - h)^2 = 4a(y - k)$ (parabola opening upward)
   4. $(x - h)^2 = -4a(y - k)$ (parabola opening downward)

   where (h,k) is the coordinate of the vertex and a is the distance from the vertex to the focus.

10. **Ellipse:** An *ellipse* is the locus of all points, in a plane the sum of whose distances from two fixed points (the foci) is a constant equal to $2a$.

The ellipse is symmetrical with respect to the X and Y axes, so an ellipse with its center at the origin and its major axis along the X axis has foci at $(\pm ae,0)$ or $(\pm c,0)$, vertices at $(\pm a,0)$, and directrices at $x = \pm a/e$; its corresponding equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

An ellipse with its center at the origin and its major axis along the Y axis has foci at $(0, \pm ae)$ or $(0, \pm c)$, vertices at $(0, \pm a)$, and directrices at $y = \pm a/e$; its corresponding equation is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

The distance from the center through the focus to the curve is always designated by a and is called the *semimajor axis.* The perpendicular distance from the midpoint of the major axis to the curve is called the *semiminor axis* and is always signified by b. The distance from the center of the ellipse to the focus is called c. The eccentricity is designated by e, which is equal to c/a.

The following relationships are true for a, b, and c in an ellipse:

$$c = \sqrt{a^2 - b^2} \quad \text{or } c^2 = a^2 - b^2$$

$$b = \sqrt{a^2 - c^2} \quad \text{or } b^2 = a^2 - c^2$$

$$a = \sqrt{b^2 + c^2} \quad \text{or } a^2 = b^2 + c^2$$

$$c < a \qquad \text{and} \qquad b < a$$

Theorem: *An equation of the second degree, in which the xy term does not exist and the coeficients of x^2 and y^2 are different but have the same sign, represents an ellipse with axes parallel to the coordinate axes.*

11. **Standard equations for ellipses:**

1. $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

2. $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

where (h,k) is the center of the ellipse, a is the length of the semimajor axis, and b is the length of the semiminor axis.

12. **General equation of an ellipse:**

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where the capital letters refer to independent constants and A and C have the same sign.

13. **Hyperbola:** A *hyperbola* is the locus of all points in a plane such that the difference of their distances from two fixed points is constant. The fixed points are the *foci* and the constant difference is $2a$.

The *transverse axis* is of length $2a$ and is the distance between the intersections (vertices) of the hyperbola with its focal axis. The *conjugate axis* is of length $2b$ and is perpendicular to the transverse axis.

The *focal chord* of a hyperbola is equal to $2b^2/a$.

A hyperbola with its center at the origin and transverse axis along the X axis has foci at $(\pm ae, 0)$ or $(\pm c, 0)$, vertices at $(\pm a, 0)$, directrices at $x = \pm a/e$, and asymptotes at $y = \pm(b/a)x$; its corresponding equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

A hyperbola with its center at the origin and transverse axis along the Y axis has foci at $(0, \pm ae)$ or $(0, \pm c)$, vertices at $(0, \pm a)$, directrices at $y = \pm a/e$, and asymptotes at $x = \pm(b/a)y$; its corresponding equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

The following relationships are true for a, b, and c in a hyperbola:

$$c^2 = a^2 + b^2$$

$$b^2 = c^2 - a^2$$

$$a^2 = c^2 - b^2$$

$$c > a$$

14. **Standard equations for hyperbolas:**

1. $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

2. $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

where (h,k) is the center of the hyperbola, a is half the length of the transverse axis, and b is half the length of the conjugate axis.

15. **General equation of a hyperbola:**

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where the capital letters refer to independent constants and A and C have different signs.

16. **Polar coordinates:** The position of any point, P, is fully determined if we know the directed distance, called the *radius vector*, and the angle that the radius vector makes with the reference line, called the *polar angle*. The radius vector is denoted by ϱ, while θ denotes the polar angle. The origin is also called the *pole*.

The polar angle is positive when measured counterclockwise and negative when measured clockwise. The radius vector is positive only when it is laid off on the terminal side of the angle. When the radius vector is laid off on the terminal side of the ray produced beyond the pole (the given angle plus 180°), a negative value is assigned the radius vector.

17. **Transformation from Cartesian to polar coordinates:**

$$x = \varrho \cos \theta \qquad y = \varrho \sin \theta$$

$$\varrho^2 = x^2 + y^2 \qquad \tan \theta = \frac{y}{x}$$

18. **Transformation from polar to Cartesian coordinates:**

$$\varrho = \sqrt{x^2 + y^2}$$

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}} \qquad \sec \theta = \frac{\sqrt{x^2 + y^2}}{x}$$

$$\sin \theta = \frac{y}{\sqrt{x^2 + y^2}} \qquad \csc \theta = \frac{\sqrt{x^2 + y^2}}{y}$$

$$\tan \theta = \frac{y}{x} \qquad \cot \theta = \frac{x}{y}$$

ADDITIONAL PRACTICE PROBLEMS

1. Find the equation of the curve that is the locus of all points equidistant from the point $(-3,-4)$ and the line $6x - 8y = -2$.

2. Find the coordinates of the center and the radius of a circle for the equation $x^2 + y^2 - 10x = -9$.

3. Find the equation of the circle that passes through points $(-4,3)$, $(0,-5)$, and $(3,-4)$.

4. Give the equation; the length of a; and the length of the focal chord for the parabola, which is the locus of all points equidistant from the point $(0,-23/4)$ and the line $y = 23/4$.

5. Reduce the equation $3x^2 - 30x + 24y + 99 = 0$ to a parabola in standard form.

6. Find the equation of the ellipse with its center at the origin, semimajor axis of length 14, and directrices $y = \pm 28$.

7. Reduce the equation $4x^2 + y^2 - 16x - 16y = 64$ to an ellipse in standard form.

8. Find the equation of the hyperbola with asymptotes at $y = \pm(4/3)x$ and vertices at $(\pm 6,0)$.

9. Find the foci, directrices, eccentricity, equations of the asymptotes, and length of the focal chord of the hyperbola $\frac{y^2}{25} - \frac{x^2}{100} = 1$.

10. Change the equation $x^2 + 2x + y^2 = 0$ from rectangular to polar coordinates.

11. Change the equation $\varrho = \tan\theta \cos\theta$ to an equation in rectangular coordinates.

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. $64x^2 + 96xy + 36y^2 + 576x + 832y + 2496 = 0$

   or

   $16x^2 + 24xy + 9y^2 + 144x + 208y + 624 = 0$

2. center (5,0), radius 4

3. $x^2 + y^2 = 25$

4. $x^2 = -23y$, $a = 23/4$, $f.c. = 23$

5. $(x - 5)^2 = -8(y + 1)$

6. $\dfrac{x^2}{(7\sqrt{3})^2} + \dfrac{y^2}{(14)^2} = 1$ or $\dfrac{x^2}{147} + \dfrac{y^2}{196} = 1$

7. $\dfrac{(x - 2)^2}{(6)^2} + \dfrac{(y - 8)^2}{(12)^2} = 1$

8. $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1$

9. foci $= (0, \pm 5\sqrt{5})$; directrices $y = \pm\sqrt{5}$; eccentricity $= \sqrt{5}$; asymptotes $x = \pm 2y$; $f.c. = 40$

10. $\varrho = -2 \cos \theta$

11. $x^2 + y^2 - y = 0$